BOUNDARY VALUE APPLICATION OF A ONE-DIMENSIONAL MAXIMUM PRINCIPLE

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THESIS

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Boundary Value Application of A

One-Dimensional Maximum Principle

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ABSTRACT

The problem considered is the application of a one-dimensional maximum principle to second order, linear differential equations of the form

$$u'' + g(x)u' + h(x)u = f(x) \text{ for } a < x < b$$

with associated general boundary conditions to obtain functions $z_1(x)$ and $z_2(x)$ such that

$$z_2(x) \leq u(x) \leq z_1(x)$$

on [a,b]. The functions f,g and h are assumed to be bounded. We wish to determine the behavior of the solution u(x) on [a,b] and also to obtain reliable numerical estimates of u.

The basic concepts in the theoretical background are expanded versions of a presentation in Protter and Weinberger [Ref. 4].

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I. ONE-DIMENSIONAL MAXIMUM PRINCIPLES

If u is a real-valued continuous mapping of a compact set X, then u(X) is compact and by the Heine-Borel Theorem, u(X) is closed and bounded. Thus, u is bounded. Then there exist points p and q in X such that $u(q) \leq u(x) \leq u(p)$ for all x in X; that is, u attains its maximum (at p) and its minimum (at q). This is a specific example of a general maximum principle. We begin our discussion with a study of some specific maximum principles; namely, where and under what conditions functions of a single real variable which satisfy differential inequalities can attain their maximum value. The approach parallels a similar investigation in Protter and Weinberger [Ref. 4].

Definition: For u = (x) in C^2 [a,b], define the linear operator L on (a,b) as follows:

$$L[u] \equiv u'' + g(x)u'$$

where g(x) is any bounded function.

Lemma 1: If u = u(x) in C^2 [a,b] satisfies the differential inequality $L[u] > 0$ on (a,b), then u cannot attain a maximum at a point interior to (a,b).

Proof: Assume, to the contrary, that u assumes a relative maximum at c, $a < c < b$. Then $u'(c) = 0$ and $u''(c) \leq 0$, hence

$$L[u]\Big|_{x=c} = u''(c) + g(c)u'(c) = u''(c) \leq 0$$

which is a contradiction of $L[u] > 0$ on (a,b). Hence u can attain a maximum only at the end points a or b. We note that the boundedness of g was necessary in the proof of Lemma 1 since, if $\lim_{x \to c} g(x) = \infty$, then $L[u]\Big|_{x=c}$ would be undefined. The boundedness condition for g may be weakened. The fact that g(x) is bounded on every closed subinterval of (a,b) will be sufficient for our purposes. It is important to note that g bounded on every closed subinterval of (a,b) does not imply that g is bounded on [a,b]. Consider g(x) = 1/(1-1) on (-1,1), and g(x) unbounded as $x \longrightarrow 1$.

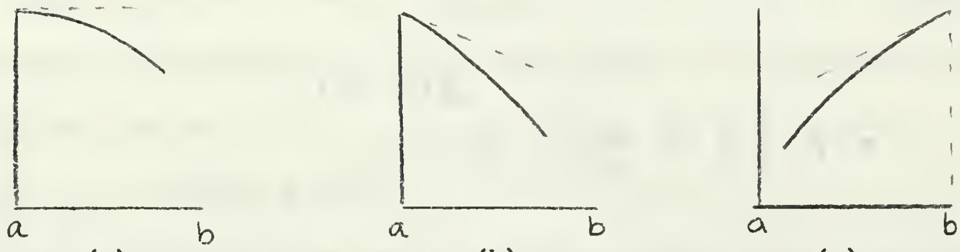
<u>Lemma</u> 2: If $v = v(x)$ in C^2 [a,b] satisfies the differential inequality $L\ [v] < 0$ on (a,b), then v cannot attain a minimum at a point interior to (a,b).

Proof: Note that -v satisfies the hypotheses of Lemma 1 since differential operators are linear; that is $L\ [-v] = -L\ [v] > 0$. Hence -v cannot attain a maximum interior to (a,b) which implies v cannot have a minimum interior to (a,b). We now have as tools two simple one-dimensional minimum and maximum principles. For convenience we will generally discuss explicitly only the maximum principles; but with reasoning similar to the proof of Lemma 2, we will be implicitly discussing the related minimum principle.

To continue our development we relax the condition of strict inequality on the differential inequality and now consider functions u such that $L\ [u]\ \ 0$. The solution u identically a constant does not satisfy the strict inequality in the hypotheses of Lemmas 1 and 2 since $L\ [u] \equiv 0$. We now allow as a special case the constant function, which attains its maximum at every point in [a,b]. We will show in a Corollary to Theorem 1 that the functions $u \equiv$ constant are the only functions which satisfy $L\ [u] \geqslant 0$ and attain their maximum interior to (a, b).

In investigating the behavior of a function u which attains its maximum at x = a, we might imagine that u could have a graph of the form shown in Fig. 1a (and similarly if the maximum is attained at x = b).

That is, u attains its maximum at $x = a$ with $u'(a) = 0$ (or at $x = b$ with $u'(b) = 0$).



(a) (b) (c)

Fig. 1

We show now that this behavior is impossible if u satisfies $L\ [u] \geqslant 0$. If M is the maximum of u on $[a,b]$, then $u(a) = M$ implies $u'(a) < 0$ (Fig. 1b): and $u(b) = M$ implies $u'(b) > 0$ (Fig. 1c).

<u>Theorem 1</u>: Let $u = u(x)$ in $C^2\ [a,b]$ be a nonconstant function which satisfies the differential inequality

$$u'' + g(x)u' \geqslant 0$$

on (a,b) and has one-sided derivatives at a and b, and suppose further that g is bounded on every closed subinterval of (a,b). Then:

(i) the maximum of u occurring at $x = a$ and g bounded below at $x = a$ imply $u'(a) < 0$;

(ii) the maximum of u occurring at $x = b$ and g bounded above at $x = b$ imply $u'(b) > 0$.

Proof: Suppose that $u(a) = M$ and $u(x) \leqslant M$ for $a < x < b$. Since u is nonconstant by hypothesis, there exists a point d in (a,b) such that $u(d) < M$. Define the function

$$z(x) = e^{A(x-a)} - 1$$

with A a positive constant to be determined. We note that $z(x)$ is positive on (a,b) and that

$$L[z] = z'' + g(x)z'(x) = A^2 e^{A(x-a)} + g(x) A e^{A(x-a)}$$

$$= A\left[A + g(x)\right] e^{A(x-a)}$$

We choose A large enough so that L [z] $>$ 0 on (a,d); that is, we select A such that it satisfies the inequality

$$A > -g(x)$$

on [a,d] which is possible by the boundedness assumptions on g(x). Now form the function

$$w(x) = u(x) + E\, z(x)$$

where E is a positive constant chosen so that it satisfies

$$E < \frac{M - u(d)}{z(d)}$$

This is possible since the numerator is positive and since u(d) $<$ M. Now

$$L[w] = L[u] + E\, L[z] > 0$$

on (a,d) and hence the maximum of w on [a,d] must occur at one of the end points by Lemma 1. We have

$$w(a) = u(a) + E\, z(a) = M$$

and

$$w(d) = u(d) + E\, z(d) < u(d) + M - u(d) = M$$

upon substituting the inequality for E; hence w(a) $>$ w(d) implies the maximum occurs at a. Obviously, the one-sided derivative of w at x = a cannot be positive for then w(a) would not be a maximum. Thus

$$w'(a) = u'(a) + E\, z'(a) \leq 0$$

and since $z'(a) = A > 0$, it follows that $u'(a) < 0$, the desired result in (i).

Suppose that $u(b) = M$ and $u(x) \leqslant M$ for $a < x < b$. Since u is nonconstant by hypothesis there exists a point d in (a,b) such that $u(d) < M$. Define

$$z(x) = e^{-A(x-b)} - 1$$

with A a positive constant to be determined. Then $z(x)$ is positive on (a,b) and

$$L[z] = z'' + g(x) z' = A^2 e^{-A(x-b)} - A g(x) e^{-A(x-b)}$$

$$= A\left[A - g(x)\right] e^{-A(x-b)}$$

We choose A large enough so that $L\ [z] > 0$ on (d,b); that is, we select A such that it satisfies the inequality

$$A > g(x)$$

on $[d,b]$ which is possible by the boundedness assumptions on $g(x)$. Now we proceed as before and form the function $w(x) = u(x) + E\ z(x)$, and we reach the conclusion that w attains a maximum at $x = b$. The one-sided derivative of w at $x = b$ cannot be negative for then $w(b)$ would not be a maximum. Then

$$w'(b) = u'(b) + E\, z'(b) \geqslant 0$$

and since $z'(b) = -A < 0$, it follows that $u'(b) > 0$, the desired result in (ii).

<u>Corollary 1A</u>: (One-Dimensional Maximum Principle) Suppose $u = u(x)$ in $C^2\ [a,b]$ satisfies the differential inequality

$u'' + g(x)\, u' \geqslant 0$ on (a,b), with $g(x)$ bounded on every closed subinterval of (a,b). If u attains the relative maximum M at an interior point c of (a,b), then $u(x) \equiv M$.

Proof: Assume u has a maximum at c in (a,b), then $u'(c) = 0$. Applying Theorem 1 to the interval (a,c) and (c,b), we conclude that u is constant. This concludes the proof. We may prove Corollary 1A independently of Theorem 1 as follows:

Assume that $u(c) = M$ and that there exists a point d in (a,b) such that $u(d) < M$. Take $d > c$ for convenience. Then define the function

$$z(x) = (x-a)^A - (c-a)^A$$

where A is a positive constant to be determined. We note that: (i) $z(x) < 0$ for $a < x < c$, (ii) $z(c) = 0$, and (iii) $z(x) > 0$ for $c < x < b$. We have

$$L[z] \equiv z'' + g(x)\, z'$$

$$= A(A-1)(x-a)^{A-2} + g(x)\, A(x-a)^{A-1}$$

$$= A\left[(A-1) + g(x)(x-a)\right](x-a)^{A-2}$$

We choose A large enough so that $L\ [z] > 0$ for $a < x < d$; i.e., we take A to satisfy

$$(A-1) + g(x)(x-a) > 0$$

or equivalently

$$A > -g(x)(x-a) + 1$$

which is possible since g is bounded. Define the function

$$w(x) = u(x) + E\, z(x)$$

where E is a constant satisfying

$$0 < E < \frac{M - u(d)}{z(d)}$$

The existence of E is guaranteed by the hypothesis on the point d and the function z. Hence on (a,c)

$$w(x) = u(x) + E\, z(x) < M$$

by (i). Also

$$w(d) = u(d) + E\, z(d) < u(d) + M - u(d) = M .$$

At the point c

$$w(c) = u(c) + E\, z(c) = M .$$

Hence w attains a maximum greater than or equal to M at an interior point of (a,d). But

$$L[w] = L[u] + E\, L[z] > 0$$

by construction of the function z, and we contradict Lemma 1. If d $<$ c, then use the function

$$z(x) = (c-a)^A - (x-a)^A$$

which satisfies (i) z(x) $>$ 0 on (a,c) (ii) z(c) = 0, and (iii) z(x) $<$ 0 on (c,b). Note that the inequalities (i) and (iii) are reversed for this case. This concludes the proof. It would also be possible to use the function

$$z(x) = e^{A(x-c)} - 1$$

as the auxiliary function in the proof of Theorem 1, if d $<$ c, and

$$z(x) = e^{-A(x-c)} - 1$$

if d $>$ c.

Corollary 1B: A nonconstant u satisfying the differential inequality $L[u] \geq 0$ on (a,b) cannot have a relative maximum at an interior point.

Proof: Assume, to the contrary, that u is nonconstant and has a relative maximum at an interior point c. Then apply Corollary 1A to a subinterval I containing c, where I is small enough that u(c) is an absolute maximum on I. This yields the immediate contradiction that $u \equiv M$.

Corollary 1C: A nonconstant function u satisfying the differential inequality $L[u] \geq 0$ on (a,b) can have at most one relative minimum in the open interval (a,b).

Proof: We note that between any minima there must be a relative maximum which contradicts Corollary 1B.

The boundedness properties of g are required for Theorem 1 and Corollary 1A as shown in the following example. Consider the functions $u(x) = \cos x$ and $g(x) = -\cot x$, which satisfy the differential equation

$$u'' + g(x)u' = -\cos x - \cot x\,(-\sin x)$$
$$= -\cos x + \cos x = 0$$

and here $u'' + g(x)\,u' \geq 0$. Conclusion (i) of Theorem 1 does not hold on $[0, \pi/2]$ since $u'(0) = -\sin(0) = 0$ and u is a nonconstant function. On $[-\pi/2, \pi/2]$ the conclusion of Corollary 1A fails to hold since $u(0) = \cos(0) = 1$ is a maximum at an interior point and u is a nonconstant function. We note that g(x) is unbounded in the neighborhood of the origin.

Definition: A function u(x) has a horizontal point of inflection at $x = c$ if $u'(c) = 0$ while u is strictly increasing or strictly decreasing in some interval containing c.

Corollary 1D: A function u satisfying the differential inequality $L[u] \geq 0$ on (a,b) can have no horizontal point on inflection in (a,b).

Proof: Assume, to the contrary, that at the point c in (a,b), $u'(c) = 0$. Then on [a,c] or [c,b], u must attain its maximum on the interval at $x = c$, a contradiction to Theorem 1.

Definition: For $u = u(x)$ in C^2 [a,b] and h(x) and bounded function, define the linear operator (L+h) on (a,b) as follows

$$(L+h)[u] \equiv u'' + g(x)u' + h(x)u$$

where g and h are any bounded functions.

In this more general setting, it is necessary to modify some of the preceeding discussion as shown by the following examples. Consider the differential equation

$$u'' + u = 0 \quad \text{on} \quad [0,\pi]$$

which has the solution $u(x) = \sin x$. Now u attains a maximum of 1 at $x = \pi/2$, an interior point of $[0,\pi]$.

We also see that the condition $h(x) \leq 0$ is not sufficient to yield an unrestricted maximum principle. Consider

$$u'' - u = 0 \quad \text{on} \quad [-1,1]$$

which has the solution $u(x) = -e^x - e^{-x}$, and u attains a negative maximum of -2 at $x = 0$. We will prove that a nonconstant function u satisfying $(L+h)[u] \geq 0$ with $h(x) \leq 0$ on (a,b) cannot attain a non-negative maximum at an interior point.

Lemma 3: If u satisfies $(L+h)[u] \geq 0$, with $h(x) \leq 0$ on (a,b), then a nonconstant function u cannot attain a non-negative maximum at an interior point.

Proof: Assume, to the contrary, that $u(c) = M$ for $a < c < b$ and $u(x) \leq M$ on (a,b). Then $h(c)u(c) \leq 0$, $u'(c) = 0$, $u''(c) \leq 0$; hence

$$(L+h)[u]\Big|_{x=c} = u''(c) + g(c)\,u'(c) + h(c)\,u(c)$$

$$= u''(c) + h(c)\,u(c) \leq 0$$

a contradiction of $(L+h)\ [u] > 0$ on (a,b).

Theorem 2: If $u = u(x)$ in $C^2\ [a,b]$ satisfies the differential inequality

$$(L+h)[u] \equiv u'' + g(x)\,u' + h(x)\,u \geq 0$$

on an interval (a,b) with $h(x) \leq 0$, if g and h are bounded on every closed subinterval, and if u attains a non-negative maximum value M at an interior point c, then $u(x) \equiv M$.

Proof: Suppose that $u(c) = M$ and $u(x) \leq M$ for $a < x < b$ and $M \geq 0$. Then assume that there exists a point d in (a,b) such that $u(d) < M$ and for convenience take $d > c$. Define the function

$$z(x) = e^{A(x-c)} - 1$$

where A is a positive constant to be determined. We note that: (i) $z(x) < 0$ on (a,c): (ii) $z(c) = 0$; and (iii) $z(x) > 0$ on (c,b). Now

$$(L+h)[z] \equiv z'' + g(x)\,z' + h(x)\,z$$

$$= A^2 e^{A(x-c)} + A g(x) e^{A(x-c)} + h(x)\left[e^{A(x-c)} - 1\right]$$

$$= e^{A(x-c)}\left[A^2 + A g(x) + h(x)\right] - h(x)$$

We choose A large enough so that (L+h) [z] $>$ 0 on (a,d), that is, so that A satisfies the inequality

$$\left[A^2 + Ag(x) + h(x)\right] - h(x)e^{-A(x-c)} > 0.$$

Since $e^{A(x-c)} > 0$ and $h \leq 0$ on (a,b), we need only to choose A such that

$$A^2 + Ag(x) + h(x) > 0$$

which is possible since g and h are bounded. Now form the function

$$w(x) = u(x) + E\,z(x)$$

where E is a positive constant so that

$$E < \frac{M - u(d)}{z(d)}$$

Hence on (a,c)

$$w(x) = u(x) + E\,z(x) < M$$

since z(x) $<$ 0 on (a,c). And by the definition of E,

$$w(d) = u(d) + M - u(d) = M$$

Now

$$w(c) = u(c) + E\,z(c) = M$$

since z(c) = 0; thus, we may conclude that w has a maximum greater than or equal to M which is attained interior to (a,d). But

$$(L+h)[w] = (L+h)[u] + E\,(L+h)[z] > 0$$

on (a,d) and we have a contradiction to Lemma 3 since w(x) is nonconstant.

If d $<$ c, construct the function

$$z(x) = e^{-A(x-c)} - 1$$

which has the properties: (i) z(x) $>$ 0 on (a,c), (ii) z(c) = 0, (iii) z(x) $<$ 0 on (c,b). Now

$$(L+h)[z] \equiv z'' + g(x) z' + h(x) z$$

$$= A^2 e^{-A(x-c)} - A g(x) e^{-A(x-c)} + h(x)\left[e^{-A(x-c)} - 1\right]$$

$$= e^{-A(x-c)}\left[A^2 - A g(x) + h(x)\right] - h(x)$$

We choose A large enough so that $(L+h)\ [z] > 0$, that is so that the following inequality is satisfied:

$$\left[A^2 - A g(x) + h(x)\right] - h(x) e^{A(x-c)} > 0$$

since $e^{-A(x-c)} > 0$ on (a,b). And since $h \leq 0$, choose A such that

$$A^2 - A g(x) + h(x) > 0$$

which is possible since g and h are bounded. We note that the condition that A satisfies

$$A^2 - A|g(x)| + h(x) > 0$$

will suffice for both $d < c$ and $d > c$. The remainder of the proof is exactly the same as before with $c < d$.

<u>Corollary 2A</u>: If u satisfies $(L+h)\ [u] \geq 0$ on (a,b) with $h \leq 0$ and $h(x) \not\equiv 0$, and u assumes a non-negative maximum value at an interior point, then $u(x) \equiv 0$.

Proof: Let $M \geq 0$ be the maximum attained by u on $[a,b]$. By Theorem 2, $u(x) \equiv M$ and hence

$$(L+h)[M] = h(x) M \geq 0$$

But $h(x) \leqq 0$ and $h(x) \not\equiv 0$ implies $M = 0$.

Theorem 3: Suppose that $u = u(x)$ in $C^2\,[a,b]$ satisfies the differential inequality

$$(L+h)[u] \equiv u'' + g(x)u' + h(x)u \geqq 0$$

on (a,b) and has one-sided derivatives at a and b, that $h \leqq 0$ and that g and h are bounded on every closed subinterval of (a,b). Then:

(i) the non-negative maximum of u occurring at $x = a$ and the function $g(x) + (x-a)h(x)$ bounded from below at $x = a$ imply that $u'(a) < 0$;

(ii) the non-negative maximum of u occurring at $x = b$ and the function $g(x) - (b-x)h(x)$ bounded from above at $x = b$ imply that $u'(b) > 0$.

Proof: We may extend the proof of Theorem 1 by noting that if the maximum of uc occurs at $x = a$, then

$$(L+h)[e^{A(x-a)}-1] = e^{A(x-a)}\left[A^2 + Ag(x) + h(x)\left(1-e^{-A(x-a)}\right)\right]$$

$$\geqq e^{A(x-a)}\left[A^2 + Ag(x) + A(x-a)h(x)\right]$$

since $e^{t} \geq \frac{1}{t+1}$ for $t \geqq 0$ and $h(x) \leqq 0$ by assumption. If $g(x) + (x-a)h(x)$ is bounded from below at $x = a$, then we may choose A such that

$$A > -\left[g(x) + (x-a)h(x)\right]$$

If the maximum of u occurs at $x = b$, then as above

$$(L+h)[e^{-A(x-b)}-1] = e^{-A(x-b)}\left[A^2 - Ag(x) + h(x)\left(1-e^{A(x-b)}\right)\right]$$

$$\geqq e^{-A(x-b)}\left[A^2 - Ag(x) + A(b-x)h(x)\right]$$

Hence if $g(x) - (b-x)h(x)$ is bounded above at $x = b$, then we may choose A such that

$$A > g(x) - (b-x)h(x)$$

The remainder of the proof is analogous to Theorem 1.

<u>Corollary 3A</u>: If $u \not\equiv 0$ satisfies $(L+h)\ [u] \geq 0$ on (a,b) with h 0, if u is in $C^2[a,b]$, and if $u(a) \leq 0$, $u(b) \leq 0$, then $u(x) < 0$ on (a,b).

Proof: Assume that there exists a point c on (a,b) such that $u(c) > 0$. Then $u(a) \leq 0$ and $u(b) \leq 0$ imply that u attains a non-negative maximum at some point d interior to (a,b), say $u(d) = M$. Then, by Theorem 2, $u(x) \equiv M \geq 0$, a contradiction.

A useful technique in determining the existence of a maximum (or minimum) of a solution to a differential equation at an interior point, independent of the preceeding theory, is a simple direct argument.

Consider the problem

$$u'' + e^{u} = -x \quad \text{on} \quad (0,1),$$

which does not have the format of the problems we have so far considered due to the exponential term e^{u}.

We note that

$$f(u) \equiv u'' + e^{u} = -x < 0$$

on $(0,1)$. Hence, u cannot attain a minimum in $(0,1)$; otherwise, if there exists a point c in $(0,1)$ such that $u(c) = M$ and $u(x) \geq M$ for $x \neq c$; then $u''(c) \geq 0$ and we have

$$f(u)\Big/_{x=c} = u''(c) + e^{u(c)} > 0$$

on $(0,1)$, a contradiction.

As another example of the preceeding, consider

$$u'' - 2\cos(u') = 1$$

whose solution u cannot attain a local maximum in the interior of any interval. Let I be an interval with subinterval [a,b]. If there exists c in (a,b) such that u attains a maximum in [a,b] at c, then

$$u''(c) \leq 0 \qquad \text{and} \qquad u'(c) = 0$$

Hence

$$0 \geq u''(c) = 1 + 2\cos[u'(c)] = 1 + 2\cos(0) = 3,$$

an impossibility.

We may even apply the previous results to boundary value problems. Consider

$$u'' + e^{x}u' = -1$$

for 0 x 1, with u(0) = u(1) = 0. Now

$$u'' + e^{x}u' < 0$$

on (0,1); hence u cannot attain a minimum interior to (0,1) as in the first example. The boundary conditions u(0) = u(1) = 0 imply that u'(0) $\geq$ 0 and u'(1) $\leq$ 0; otherwise u would attain a minimum interior to (0,1). If u'(0) = 0 and u(0) = 0, then from the differential equation, u"(0) = -1. The boundary condition u(1) = 0 forces u to attain a minimum interior to (0,1), a contradiction. We obtain a similar result if we assume u(1) = u'(1) = 0, since u(0) = 0. Thus u'(0) $>$ 0 and u'(1) $<$ 0.

II. A GENERALIZED ONE-DIMENSIONAL MAXIMUM PRINCIPLE

If we drop the restriction that the function h must be non-positive we may then obtain a maximum principle for $(L+h)\ [u] \geq 0$ by introducing a function w, with certain desirable properties, such that the quotient u/w satisfies the previous maximum principles.

Theorem 4: Suppose $u = u(x)$ in $C^2\ [a,b]$ satisfies the differential inequality $(L+h)\ [u] \geq 0$ on (a,b) with $h(x)$ bounded and $g(x)$ bounded below; then for any sufficiently small subinterval $[a',b']$, there exists a function w in $C^2\ [a,b]$ such that:

(i) $w > 0$ on $[a',b']$

(ii) $(L+h)\ [w] \leq 0$ on (a',b').

Then the function u/w satisfies the maximum principles of Theorems 2 and 3 on (a',b').

Proof: Consider the function

$$w(x) = 1 - B(x-a')^2$$

for $a < a' < x < b' < b$, where B is a positive constant to be determined. When B has been determined, we will suppose that $(b'-a')$ is such that

(1) $$B(b'-a')^2 < 1$$

so that $w > 0$ on $[a',b']$. Now

$$\begin{aligned}(L+h)[w] &\equiv w'' + g(x)w' + h(x)w \\ &= (-2B) + g(x)\left[-2B(x-a')\right] + h(x)\left[1 - B(x-a')^2\right] \\ &= -2B\left[1 + (x-a')g(x) + \tfrac{1}{2}(x-a')^2 h(x)\right] + h(x)\end{aligned}$$

Since g and h are bounded below, choose constants G and H such that $g \geqslant G$ and $h \geqslant H$. We now make a restriction on (b',a') so that

$$(2) \quad 1 + (x-a')G + \tfrac{1}{2}(x-a')^2 H > 0$$

for $a' \leq x \leq b'$ and choose B so that $(L+h)[w] \leq 0$ on (a',b'), i.e.,

$$B \geq \frac{1}{2}\left[\frac{h(x)}{1+(x-a')G+(x-a')^2H}\right]$$

a permissible choice since h is bounded above. Hence if the subinterval [a',b'] is sufficiently small, the function w with properties (i) and (ii) may always be constructed.

Define the function v = u/w on [a',b']; then

$$(L+h)[u] = (L+h)[vw] = (vw)'' + g(x)(vw)' + h(x)vw$$

$$= (vw'' + 2v'w' + wv'') + g(x)(v'w + w'v) + h(x)vw$$

Now dividing by the positive function w, we obtain

$$v'' + \left[2\frac{w'}{w} + g(x)\right]v' + \frac{1}{w}(L+h)[w]\,v \geq 0.$$

Thus, v satisfies the differential inequality

$$v'' + g_1(x)v' + h_1(x)v \geq 0$$

on (a',b') where

$$g_1(x) = 2\frac{w'}{w} + g(x); \quad h_1(x) = \frac{1}{w}(L+h)[w].$$

By the construction of w, w in C^2 [a',b']; thence g and h are bounded on [a',b']. Also h is non-positive by conditions (i) and (ii). Thus v satisfies the hypotheses for Theorems 2 and 3 on subinterval [a',b'], which concludes the proof.

We now see that a function u which satisfies $(L+h)[u] \geq 0$ has certain restrictions on its zeros on the interval under consideration.

Corollary 4A: If the function u satisfies Theorem 4 on (a',b') and $u(x) \not\equiv 0$, then u can have at most two zeros (between which u is negative) on (a',b').

Proof: We may take the interval $[a',b']$ such that $u(a') = u(b') = 0$ and $u(x) \neq 0$ for $a' < x < b$. Assume that $u(x) > 0$ for $a' < x < b$. Then $v(x) = u(x)/w(x) > 0$ for $a' < x < b'$ and $v(a') = v(b') = 0$. Hence v has a non-negative maximum in (a',b') which is a contradiction to Theorem 4. Thus, $u(x) < 0$ on (a',b'). Also, there could not be more than two zeros; otherwise, if there exists a point c in (a',b') such that $u(c) = 0$, then $u(x) < 0$ on (a',b') forces $u'(c) = 0$. But then u must be the trivial solution, a contradiction.

Note that the zeros of u must be isolated; i.e., if $u(a) = 0$, then u cannot vanish on some finite interval to the right of a.

Corollary 4B: If u is a solution to the differential equation $(L+h)[u] = 0$, then u can have at most one zero in any interval (a',b') where Theorem 4 holds.

Proof: Note that -u also satisfies $(L+h)[-u] = 0$. By Corollary 4A, u and -u can each have at most two zeros on (a',b'), between which both u and -u must both be negative, an impossibility. Thus, u has at most one zero on (a',b').

We may conclude that the boundedness hypothesis in Theorem 4 on the function g is essential by noting the following example. The function $u(x) = x$ satisfies $u'' + g(x)u' = 0$ with $g(x) = -2/x$, and g is unbounded in the neighborhood of zero. Consider the interval of the form (a,b) with $b = 0$. Then u attains a non-negative maximum of zero at $x = b$ and

$u'(b) = 0$, contradicting conclusion (ii) of Theorem 3, hence contradicting Theorem 4.

Definition: Let $r(x)$ be a solution of the differential equation

$$r'' + g(x)r' + h(x)r = 0$$

with g and h bounded functions. If $r \not\equiv 0$ and $r(a) = 0$, the r must be non-zero on some finite interval to the right of a by Corollary 4B. If r has another zero on (a, ∞), denote the first such zero by a^* and call it the conjugate point of a. (If $r \neq 0$ on (a, ∞), then set $a^* = \infty$).

Thus r has one sign on (a,a^*) and for convenience we may take $r(x) > 0$ for $a < x < a^*$, since $-r$ satisfies the same differential equation. We now establish a necessary and sufficient condition for the existence of the function w of Theorem 4 in terms of a^*.

Corolllary 4C: If a^* is the conjugate point of a, there exists a function $w > 0$ such that Theorem 4 holds on the interval $[a,b]$ if and only in $b < a^*$.

Proof: Assume that $b > a^*$. If $w > 0$ on $[a,a^*]$, then the function u/w vanishes at a and a^* since $r(a) = r(a^*) = 0$, and u/w is positive on (a,a^*). Hence u/w attains a non-negative maximum in (a,a^*) and u/w is non-constant. then w cannot satisfy $(L+h)\,[w] \leq 0$ on (a,a^*) without contradicting Theorem 4. Hence by a contrapositive argument, we have the forward implication that the existence of an acceptable $w(x)$ implies $b < a^*$.

To get the reverse implication, let b in (a,a^*). Note that $r(x)$ is bounded from below by a positive number $m = m(c)$ on any subinterval $[c,b]$ with $a < c$. Let

$$z(x) = 2 - e^{A(x-a)}$$

where A is a positive constant to be determined. Consider the function

$$w(x) = v(x) + E\, z(x)$$

Now choose E small enough so that w $>$ 0 on [a,b]. Now

$$(L+h)[w] = \cancel{(L+h)[v]}^{\,0} + E(L+h)[2 - e^{A(x-a)}]$$

$$= E\left\{(-A^2 e^{A(x-a)}) + g(x)(-Ae^{A(x-a)}) + h(x)(2-e^{A(x-a)})\right\}$$

$$= -Ee^{A(x-a)}\left\{A^2 + Ag(x) + h(x)\left[1 - 2e^{-A(x-a)}\right]\right\}$$

We want to choose A such that (L+h) [w] $\leq$ 0 on [a,b] which can be done as follows:

Now $-Ee^{A(x-a)} < 0$ since E and $e^{A(x-a)}$ are positive. We make two restrictions on A:

(i) $1 - 2e^{-A(x-a)} > 0$, i.e., $\quad A > \dfrac{\ln 2}{x-a}$

Since h is bounded, we may also choose A large enough so that

(ii) $\quad A^2 + Ag(x) + h(x)\left[1 - 2e^{-A(x-a)}\right] > 0.$

Then we have (L+h) [w] $\leq$ 0, and w is a function for which Theorem 4 holds.

If h(x) is unbounded or g(x) is not bounded below there may be no interval [a',b'] for which Theorem 4 holds. Consider the function

$$u(x) = x \sin\left(\frac{1}{x}\right)$$

which satisfies

$$u'' + x^{-4} u = 0$$

Now as $x \to 0+$, $h(x) = x \to +\infty$. There can be no function w $>$ 0 with the property that u/w satisfies the maximum principles in any of the intervals $[0, 1/n\pi]$ (n = 1,2,...) since $u(1/n\pi) = 0$ and $1/n\pi \to 0$ as $n \to \infty$. Thus, for all $\varepsilon > 0$, the interval $(0, \varepsilon)$ contains a zero of u and Corollary 4A denies the existence of an appropriate w(x).

As in the examples in the previous section, we use a direct argument to conclude that no solution of

$$u'' + e^{u} = e$$

can attain a minimum value greater than 1 or a maximum value less than 1.

Assume there exists a point c such that:

(i) $u(c)$ = maximum; hence $u''(c) \leq 0$ and we have

$$e^{u(c)} - e = -u''(c) \geq 0$$

which implies $e^{u(c)} \geq e$, and we have the maximum value of u is $u(c) \geq \ln(e) = 1$;

(ii) $u(c)$ = minimum; hence $u''(c) \geq 0$ and we have

$$e^{u(c)} - e = u''(c) \leq 0$$

which implies that $e^{u(c)} \leq e$ and we have the minimum value of u is $u(c) \leq \ln(e) = 1$.

III. INITIAL VALUE PROBLEM

Consider the solution to the differential equation

$$(1) \qquad u'' + g(x)u' + h(x)u = f(x)$$

which satisfies the initial conditions

$$(2) \qquad u(a) = \gamma_1 \; ; \quad u'(a) = \gamma_2$$

where the functions f, g and h are defined on the interval (a,b) with g and h bounded and γ_1 and γ_2 given constants.

The existence of a solution follows from the classical theory of ordinary differential equations [Ref. 3]. We may prove the uniqueness of such a solution independent of the classical theory using Theorem 4.

Theorem 5: Suppose the $u_1(x)$ and $u_2(x)$ are solutions of (1) which satisfy the same initial conditions (2) in an interval (a,b). Then $u_1 \equiv u_2$ in (a,b).

Proof: Define $u(x) = u_1(x) - u_2(x)$. Then u satisfies

$$u'' + g(x)u' + h(x)u = 0$$

with initial conditions

$$u(a) = u'(a) = 0$$

Now assume that $u \not\equiv 0$ in (a,b). Then by Theorem 4 there exists a positive function w on an interval $(a, a+\varepsilon)$ such that the maximum of u/w in $(a, a+\varepsilon)$ must occur at one of the endpoints. We note that is a positive constant depending only on the bounds of g and h. I.e., $(a+\varepsilon) - \varepsilon = \varepsilon$ must satisfy conditions (1) and (2) in the proof of Theorem 4. But -u also satisfies the same equation with the same initial conditions, hence, also by Theorem 4, -u/w must also attain its maximum at one of the endpoints, a or a+. Hence u/w must attain either a maximum or a minimum at a. But at x = a

$$\left(\frac{u}{w}\right)'\Bigg|_{x=a} = \frac{u'w - uw'}{w^2}\Bigg|_{x=a} = 0$$

since u(a) = u'(a) = 0. Now the function u/w satisfies Theorem 3; hence u/w must be identically a constant, and from the initial condition, $u \equiv 0$ on $[a, a+\varepsilon]$. In particular

$$u(a+\varepsilon) = u'(a+\varepsilon) = 0$$

Hence, by the analogous argument, $u \equiv 0$ on $[a+\varepsilon, a+2\varepsilon]$ with ε unchanged since the choice of ε is independent of u. Now $(b-a) < \infty$; hence, after a finite number of steps, we may conclude that $u \equiv 0$ on [a,b]. Thus $u_1 \equiv u_2$.

Consider the problem

$$u'' - \frac{1}{x}u' = 0$$

with the initial conditions

$$u(0) = u'(0) = 0$$

We note that $u_1 \equiv 0$ satisfies the equations trivially and for $u_2 = x^2$,

$$u_2'' - \frac{1}{x}u_2' = 2 - \frac{1}{x}(2x) \equiv 0.$$

But $u_1 \not\equiv u_2$ on (0,1). Since $g(x) \longrightarrow \infty$ as $x \longrightarrow 0$, we may conclude that the boundedness of g is necessary in Theorem 5.

We can easily find a function h(x) such that the equation

$$u'' + h(x)u = 0$$

has two solutions satisfying

$$u(0) = u'(0) = 0$$

Obviously $u_1 \equiv 0$ is a solution. We must then pick h(x) unbounded in the neighborhood of the origin; otherwise Theorem 5 would imply that the trivial solution was the only solution. Choose $h(x) = -6/x^2$, then $u(x) = x^3$ is an appropriate solution satisfying the initial conditions.

Hence we see that the boundedness hypothesis on h is also necessary in Theorem 5.

An interesting application of the Mean Value Theorem arises in the initial value problem

(3) $$u'' + e^u = 1 \quad \text{for } x > 0$$

satisfying

(4) $$u(0) = 1 \quad \text{and} \quad u'(0) = 0.$$

Assume that u_1 and u_2 are solutions of the differential equation (3) satisfying the same initial conditions (4). Then the function

$$u(x) = u_1(x) - u_2(x)$$

satisfies the homogeneous differential equation

$$(u_1'' - u_2'') + (e^{u_1} - e^{u_2}) = 0.$$

By use of the Mean Value Theorem

$$e^{u_1} - e^{u_2} = (u_1 - u_2)\, e^{u_2 + \theta(u_1 - u_2)} \quad ; \; 0 < \theta < 1$$

and the differential equation becomes

$$(u_1'' - u_2'') + (u_1 - u_2)\, e^{u_2 + \theta(u_1 - u_2)} = 0$$

with

$$u(0) = u'(0) = 0$$

Since $h(x) = -e^{u_2 + \theta(u_1 - u_2)} \leq 0$ and is bounded on any finite subinterval of $(0, \infty)$, we may apply Theorem 5. The trivial solution $u \equiv 0$ satisfies the above initial value problem, thus u is unique and $u_1 \equiv u_2$. Hence we have the solution of (3) satisfying (4).

IV. BOUNDARY VALUE PROBLEM

We will first be concerned with boundary value problems of the type

$$u'' + g(x)u' + h(x)u = f(x) \tag{1}$$

on the interval (a,b) subject to the boundary conditions

$$u(a) = \gamma_1 \ ; \ u(b) = \gamma_2 \ . \tag{2}$$

We may derive a uniqueness theorem by the use of Theorem 4 for solutions of (1) which satisfy (2). The sign of the function h(x) has great significance as seen in the simple example:

$$u'' + u = 0$$

which has the solutions $u_1 \equiv 0$ and $u_2 = \sin x$ on the interval $[0, \pi]$. Both u_1 and u_2 satisfy the boundary conditions $u(0) = u(\pi) = 0$.

<u>Theorem 6</u>: Suppose that $u_1(x)$ and $u_2(x)$ are solutions of (1) which satisfy the boundary conditions (2). If $h(x) \leq 0$ on (a,b), then $u_1 \equiv u_2$.

Proof: Set $u(x) = u_1(x) - u_2(x)$; then u satisfies

$$u'' + g(x)u' + h(x)u = 0$$

with the boundary conditions

$$u(a) = u(b) = 0$$

By Corollary 2A, $u(x) \leq 0$ on (a,b). By analogous reasoning $-u$ satisfies the above equations; hence $-u \leq 0$ on (a,b). Thus $u \equiv 0$ on (a,b).

In order to incorporate a larger class of boundary value problems we consider more general boundary conditions which have (2) as a special case. We consider solutions of (1) which satisfy the boundary conditions

$$(3)\quad \begin{cases} -u'(a)\cos\theta + u(a)\sin\theta = \gamma_1 \\ u'(b)\cos\phi + u(b)\sin\phi = \gamma_2 \end{cases}$$

where θ, ϕ, γ_1 and γ_2 are prescribed constants with $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \frac{\pi}{2}$. We have (2) when $\theta = \phi = \pi/2$.

To derive the general boundary conditions (3) consider the boundary conditions

$$A_1(-u'(a)) + B_1 u(a) = C_1$$

$$A_2 u'(b) + B_2 u(b) = C_2$$

where A_i, B_i, C_i, $(i = 1,2)$ are given constants. We may normalize the above, i.e.,

$$\frac{A_1}{\sqrt{A_1^2+B_1^2}}(-u'(a)) + \frac{B_1}{\sqrt{A_1^2+B_1^2}}u(a) = \frac{C_1}{\sqrt{A_1^2+B_1^2}} = \gamma_1$$

and

$$\frac{A_2}{\sqrt{A_2^2+B_2^2}}u'(b) + \frac{B_2}{\sqrt{A_2^2+B_2^2}}u(b) = \frac{C_2}{\sqrt{A_2^2+B_2^2}} = \gamma_2$$

Then we define the angles θ and ϕ so that

$$\cos\theta = \frac{A_1}{\sqrt{A_1^2+B_1^2}}, \quad \sin\theta = \frac{B_1}{\sqrt{A_1^2+B_1^2}}$$

and

$$\cos\phi = \frac{A_2}{\sqrt{A_2^2+B_2^2}}, \quad \sin\phi = \frac{B_2}{\sqrt{A_2^2+B_2^2}}$$

Then $0 \leq \theta \leq \pi/2$ and $0 \leq \phi \leq \pi/2$.

Theorem 7: Suppose that $u_1(x)$ and $u_2(x)$ are solutions of (1) which satisfy boundary conditions (3). If $h(x) \leq 0$ on (a,b), then $u_1 \equiv u_2$ unless $h \equiv 0$, $\theta = \phi = 0$; in which case u_1 and u_2 may differ by a constant.

Proof: As before set $u = u_1 - u_2$, so that u satisfies

$$u'' + g(x)u' + h(x)u = 0$$

and

$$(4) \quad -u'(a)\cos\theta + u(a)\sin\theta = 0$$

$$(5) \quad u'(b)\cos\phi + u(b)\sin\phi = 0$$

Consider $u \equiv M$, a non-zero constant. Then (4) and (5) imply

$$M\sin\theta = M\sin\phi = 0$$

Now $M \neq 0$; hence $\sin\theta = \sin\phi = 0$ and $\theta = \phi = 0$. By Corollary 2A, $M \neq 0$ if and only if $h \equiv 0$. Thus, if u_1 and u_2 differ by a constant, then $h \equiv 0$ and $\theta = \phi = 0$.

Conversely, assume $\theta = \phi = 0$ and $h \equiv 0$; then $u \equiv M$ satisfies (4) and (5) trivially and

$$u'' + g(x)u' + h(x)u \equiv 0.$$

Hence, u_1 and u_2 differ by a constant.

Next assume u is a non-constant function which is positive at some point in [a,b]. By Theorem 2, u attains its positive maximum at a or b. Suppose the maximum occurs at x = a. Then by Theorem 3, $u'(a) < 0$. Now $u(a) > 0$ and $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \frac{\pi}{2}$; hence

$$-u'(a)\cos\theta + u(a)\sin\theta > 0,$$

since $\cos\theta$ and $\sin\theta$ cannot vanish simultaneously and are positive on the interval under consideration. This inequality is a contradiction of (4). If the maximum occurs at x = b, we have $u'(b) > 0$, $u(b) > 0$, and

$$u'(b)\cos\phi + u(b)\sin\phi > 0$$

a contradiction of (5). Hence u is a non-constant solution which can never be positive. Analogously -u is also a solution which satisfies (4) and (5); and we may conclude that u can never be negative. Thus $u \equiv 0$ on [a,b].

Now let the only assumption on the function h(x) be that it is bounded on the interval under consideration; i.e., we remove the restriction that h(x) be non-positive. We will now prove a uniqueness theorem, but as the following example shows, the interval considered is of vital importance. The simple equation

$$u'' + u = 0$$

with $u(a) = u(b) = 0$ has only the trivial solution for $b-a < \pi$; and for $b-a = \pi$ uniqueness failed as was shown at the beginning of this section.

<u>Theorem 8</u>: Suppose that $u_1(x)$ and $u_2(x)$ are solutions of (1) which satisfy the same boundary conditions (2) with $h(x) \leq 0$ on $[a,b]$. If $b < a^*$, where a^* is the conjugate point of a, then $u_1 \equiv u_2$.

Proof: Let $u = u_1 - u_2$; then u satisfies the equation

$$u'' + g(x)u' + h(x)u = 0$$

with

$$u(a) = u(b) = 0.$$

Since $b < a^*$ by hypothesis, Corollary 4C implies that there exists a function $w > 0$ on $[a,b]$. Hence by Theorem 4 the function $v = u/w$ satisfies the maximum principles of Theorems 2 and 3 and their Corollaries. Now $v(a) = v(b) = 0$ since $u(a) = u(b) = 0$, hence $v(x) \leq 0$ on $[a,b]$; otherwise, v would attain a positive maximum in (a,b), a contradiction to Theorem 2 and Corollary 2A. Since w is a positive function, $u(x) \leq 0$ on $[a,b]$. We may apply the same argument to $-u$ since it satisfies the same differential equation and boundary conditions, and we conclude that $-u \leq 0$. Thus $u \equiv 0$.

V. APPROXIMATION IN BOUNDARY VALUE PROBLEMS

We are concerned with the same boundary value problem as in Section IV. Thus, we seek a solution of the differential equation

$$(1) \quad (L+h)[u] \equiv u'' + g(x)u' + h(x)u = f(x)$$

on the interval (a,b) which satisfies the boundary conditions

$$(2) \quad u(a) = \gamma_1 \; ; \; u(b) = \gamma_2 .$$

Even in many of the simplest problems an explicit expression of the solution is impossible to determine. We must then resort to series solution techniques or other essentially numerical methods. Obviously an approximation solution technique which yields an explicit bound for the error compared to the exact solution is desirable. It is possible to find such a bound by finding explicit approximation functions which provide an upper and lower bound for the actual values of the solution.

In this section we will develop necessary conditions for the existence of such approximation functions, and in Section VI we will investigate more closely the approximation functions and related results.

We assume, as before, that the functions f, g and h are bounded on the interval under consideration. With these assumptions it is possible to determine bounds for a solution u of (1) and (2) using the maximum principle in Theorem 3 without any further knowledge about u.

Theorem 9: Suppose that u is a solution of (1) satisfying boundary conditions (2), and that $h(x) \quad 0$. If the function $z_1(x)$ satisfies:

$$(3) \quad (L+h)[z_1] \leq f(x) \quad \text{on} \quad (a,b)$$

(4) $z_1(a) \geq \gamma_1$ and $z_1(b) \geq \gamma_2$

and the function $z_2(x)$ satisfies:

(5) $(L+h)[z_2] \geq f(x)$ on (a,b)

(6) $z_2(a) \leq \gamma_1$ and $z_2(b) \leq \gamma_2$

then

$$z_2(x) \leq u(x) \leq z_1(x)$$

for $a \leq x \leq b$.

Proof: Consider the function

$$v_1(x) \equiv u(x) - z_1(x)$$

Then v_1 satisfies

$$(L+h)[v_1] \geq 0$$

on (a,b) with

$$v_1(a) \leq 0 \text{ and } v_1(b) \leq 0.$$

Apply Corollary 3A; $v_1(x) \leq 0$ on [a,b]. Thus,

$$u(x) \leq z_1(x).$$

Similarly the function

$$v_2(x) = z_2(x) - u(x)$$

satisfies

$$(L+h)[v_2] \geq 0$$

on (a,b) with

$$v_2(a) \leq 0 \text{ and } v_2(b) \leq 0.$$

Thus $v_2(x) \leq 0$ on [a,b] and we have

$$z_2(x) \leq u(x) \leq z_1(x)$$

on [a,b].

A more general two-point boundary value problem is

(7) $(L+h)[u] = u'' + g(x)u' + h(x)u = f(x)$

on (a,b) where the solution u satisfies

(8)
$$-u'(a)\cos\theta + u(a)\sin\theta = \gamma_1$$
$$u'(b)\cos\phi + u(b)\sin\phi = \gamma_2$$

The quantities θ and ϕ are pre-assigned constants wuch that

$$0 \le \theta \le \pi/2 \quad \text{and} \quad 0 \le \phi \le \pi/2 .$$

Continuing to assume $h(x) \leq 0$, we may extend the proof of Theorem 9 to get the following.

Theorem 10: Suppose u is a solution of (7) satisfying boundary conditions (8). Suppose also that $h(x) \leq 0$ and θ, ϕ are in $[0, \pi/2]$ and that not all the equalities $h \equiv 0$, $\theta = 0$, $\phi = 0$ hold. If the function $z_1(x)$ satisfies:

$$(9) \quad (L+h)[z_1] \le f(x) \quad \text{on} \quad (a,b)$$

$$(10) \quad \begin{aligned} -z_1'(a)\cos\theta + z_1(a)\sin\theta &\ge \gamma_1 \\ z_1'(b)\cos\phi + z_1(b)\sin\phi &\ge \gamma_2 \end{aligned}$$

and the function $z_2(x)$ satisfies:

$$(11) \quad (L+h)[z_2] \ge f(x) \quad \text{on} \quad (a,b)$$

$$(12) \quad \begin{aligned} -z_2'(a)\cos\theta + z_2(a)\sin\theta &\le \gamma_1 \\ z_2'(b)\cos\phi + z_2(b)\sin\phi &\le \gamma_2 \end{aligned}$$

then

$$z_2(x) \le u(x) \le z_1(x)$$

Proof: Set the function $v_1 \equiv u - z_1$. Then v_1 satisfies

$$(L+h)[v_1] \ge 0$$

and

$$-v_1'(a)\cos\theta + v_1(a)\sin\theta \le 0$$

$$v_1'(b)\cos\phi + v_1(b)\sin\phi \le 0 .$$

If v_1 is always positive, then Theorem 2 states that the positive maximum of v_1 occurs at either a or b. If it occurs at a, then $v_i'(a) \leq 0$ and $v_1(a) > 0$. But

$$-v_1'(a)\cos\theta + v_1(a)\sin\theta \leq 0$$

can occur only if $\sin\theta = 0$, i.e., $\theta = 0$ and $v'(a) = 0$. Then Theorem 3 states that $v_1(x)$ is identically a positive constant. Thus $h(x) \equiv 0$ by Corollary 2A. Similarly v_1 cannot attain a positive maximum at b unless $\phi = 0$ and $h \equiv 0$, since

$$v_1'(b)\cos\phi + v_1(b)\sin\phi \leq 0.$$

We may conclude that unless $\theta = 0$, $\phi = 0$ and $h \equiv 0$, $v_1(x) \leq 0$ on (a,b); i.e.,

$$u(x) \leq v_1(x).$$

By an analogous argument, the function $v_2 \equiv z_2 - u$ satisfies

$$(L_1+h)[v_2] \geq 0 \quad \text{on} \quad (a,b)$$

and

$$-v_2'(a)\cos\theta + v_2(a)\sin\theta \leq 0$$

$$v_2'(b)\cos\phi + v_2(b)\sin\phi \leq 0.$$

If either $h \not\equiv 0$ or not both θ and ϕ are zero, then we may conclude

$$z_2(x) \leq u(x) \leq z_1(x)$$

on [a,b].

Remark: If the special case $h \equiv 0$, $\theta = \phi = 0$ holds, then from the proof of Theorem 10 we have that the functions v_1 and v_2 are both identically positive constants, say

$$v_1(x) \equiv M_1 > 0 \quad \text{and} \quad v_2(x) \equiv M_2 > 0.$$

Then

$$z_1(x) < z_1(x) + M_1 = u(x) = z_2(x) - M < z_2(x);$$

i.e., we have

$$z_1(x) < u(x) < z_2(x)$$

and z_1 and z_2 differ by a positive constant.

We drop the restrictions that $h \leq 0$ and that θ, ϕ are in $[0, \ /2]$. From the discussion on the background of the generalized Sturm-Liouville boundary conditions we may have just as easily taken θ, ϕ in $(-\pi/2.\pi/2]$ with no loss of generality.

Theorem 11: Suppose that u is a solution of (7) satisfying boundary condition (8) with $-\pi/2 < \theta \leq \pi/2$, $-\pi/2 < \phi \leq \pi/2$. Let the functions z_1 and z_2 satisfy the inequalities (9), (10) and (11), (12) respectively as in Theorem 10. Then the bounds

$$z_2(x) \leq u(x) \leq z_1(x)$$

hold in (a,b) if and only if there exists a positive function w on [a,b] satisfying

$$(13) \qquad (L+h)[w] \leq 0 \quad \text{on} \quad (a,b)$$

$$(14) \qquad \begin{aligned} -w'(a)\cos\theta + w(a)\sin\theta &\geq 0 \\ w'(b)\cos\phi + w(b)\sin\phi &\geq 0 \end{aligned}$$

in such a way that not all the inequalities in (13) and (14) are equalities.

Proof: Suppose there exists a positive function w on [a,b] which satisfies the inequalities (13) and (14). Then define the function $v = u/w$. As in Theorem 4

$$\begin{aligned} (L+h)[u] &= (L+h)[vw] \\ &= v'' + \left[2\frac{w'}{w} + g(x)\right]v' + \left[\frac{1}{w}(L+h)[w]\right]v = \frac{f}{w} \end{aligned}$$

which we may rewrite as

$$(15) \qquad (\bar{L}+H)[v] = v'' + Gv' + Hv = \frac{f}{w}$$

where $G = (2w'/w) + g$ and $H = 1/w(L+h)\ w$. Note that H, the coefficient of v, is non-positive. The functions v satisfies the boundary conditions:

$$-\left[v'(a)\,w(a) + v(a)\,w'(a)\right]\cos\theta \;+ v(a)\,w(a)\sin\theta = \gamma_1$$

$$\left[v'(b)\,w(b) + v(b)\,w'(b)\right]\cos\phi \;+ v(b)\,w(b)\sin\phi = \gamma_2$$

which we may rearrange to form

$$-v'(a)w(a)\cos\theta + v(a)\left[-w'(a)\cos\theta + w(a)\sin\theta\right] = \gamma_1$$

(16)

$$v'(b)w(b)\cos\phi + v(b)\left[w'(b)\cos\phi + w(b)\sin\phi\right] = \gamma_2$$

Dividing through by $w(a)\cos\theta$ and $w(b)\cos\phi$, then multiplying by the non-negative quantities $\cos\bar{\theta}$ and $\cos\bar{\phi}$ in the first and second conditions of (16) respectively, we have

$$-v'(a)\cos\bar{\theta} + v(a)\left[\frac{\cos\bar{\theta}\,(-w'(a)\cos\theta + w(a)\sin\theta)}{w(a)\cos\theta}\right] = \frac{\gamma_1\cos\bar{\theta}}{w(a)\cos\theta}$$

(17)

$$v'(b)\cos\bar{\phi} + v(b)\left[\frac{\cos\bar{\phi}\,(w'(b)\cos\phi + w(b)\sin\phi)}{w(b)\cos\phi}\right] = \frac{\gamma_2\cos\bar{\phi}}{w(b)\cos\phi}$$

where the angles $\bar{\theta}$ and $\bar{\phi}$ are defined by

$$\sin\bar{\theta} = \frac{\cos\bar{\theta}\,(-w'(a)\cos\theta + w(a)\sin\theta)}{w(a)\cos\theta}$$

$$\sin\bar{\phi} = \frac{\cos\bar{\phi}\,(w'(b)\cos\phi + w(b)\sin\phi)}{w(b)\cos\phi}$$

or equivalently

$$(18)\qquad \begin{aligned} \tan\bar{\theta} &= \frac{-w'(a)\cos\theta + w(a)\sin\theta}{w(a)\cos\theta} \\ \tan\bar{\phi} &= \frac{w'(b)\cos\phi + w(b)\sin\phi}{w(b)\cos\phi} \end{aligned}$$

By the hypotheses on θ, ϕ and w, both $\tan\bar{\theta}$ and $\tan\bar{\phi}$ are non-negative. By (14) we may always take $\bar{\theta}$ and $\bar{\phi}$ such that

$$0 \le \bar{\theta} \le \pi/2 \quad \text{and} \quad 0 \le \bar{\phi} \le \pi/2 .$$

Note: If $\theta = \pi/2$, take $\bar{\theta} = \pi/2$; if $\phi = \pi/2$ take $\bar{\phi} = \pi/2$.

Hence we have

$$(19)\qquad \begin{aligned} -v'(a)\cos\bar{\theta} + v(a)\sin\bar{\theta} &= \bar{\gamma}_1 \\ v'(b)\cos\bar{\phi} + v(b)\sin\bar{\phi} &= \bar{\gamma}_2 \end{aligned}$$

with $\bar{\theta}$, $\bar{\phi}$ in $[0, \pi/2]$ and $\bar{\gamma}_1 = (\gamma_1 \cos\bar{\theta})/w(a)\cos\theta$ and $\bar{\gamma}_2 = (\gamma_2 \cos\bar{\phi})/w(b)\cos\phi$.

If the functions z_1 and z_2 satisfy the conditions (9), (10) and (11), (12) respectively; then the functions z_1/w and z_2/w satisfy the analogous conditions with respect to equation (15). Hence by Theorem 10 the following inequality holds

$$\frac{z_2(x)}{w(x)} \le \frac{u(x)}{w(x)} \le \frac{z_1(x)}{w(x)}$$

unless $\bar{\theta} = \bar{\phi} = 0$ and $H \equiv 0$. From the proof of Theorem 10 this situation holds when the inequalities (9), (10) and (11), (12) are equations. Hence, if there is a positive function w(x) satisfying (13) and (14) but such that not all are equations, we may conclude

$$(20)\qquad z_2(x) \le u(x) \le z_1(x)$$

Note: If w satisfies the boundary value problem

$$(L+h)[w] = 0 \quad \text{on } (a,b)$$

$$\begin{cases} -w'(a)\cos\theta + w(a)\sin\theta = 0 \\ w'(b)\cos\phi + w(b)\sin\phi = 0 \end{cases}$$

then let ξ be any real number and u a solution of (7) which satisfies (8). Then

$$(L+h)[u+\xi w] = (L+h)[u] + \xi(L+h)[w] = 0$$

and

$$-[u'(a)+\xi w'(a)]\cos\theta + [u(a)+\xi w(a)]\sin\theta = \gamma_1$$

$$[u'(b)+\xi w'(b)]\cos\phi + [u(b)+\xi w(b)]\sin\phi = \gamma_2 .$$

Thus, if w satisfies (13) and (14) as equations, then any multiple of w, when added to a solution of (7) and (8), will yield another distinct solution. Hence, if at least one solution u exists, then there are many.

Now consider the boundary value problem

$$(21) \qquad (L+h)[w] = 0 \quad \text{on } (a,b)$$

with

$$(22) \qquad \begin{cases} -w'(a)\cos\theta + w(a)\sin\theta = 1 \\ w'(b)\cos\phi + w(b)\sin\phi = 1 \end{cases}$$

By applying the classical existence theorems, let w be a solution of (21) satisfying (22). Noting that the function w satisfies analogous conditions to the function u in Theorem 10, we have

$$z_2(x) \leq w(x) \leq z_1(x)$$

for appropriately chosen bounding functions $z_1(x)$ and $z_2(x)$. Also $z_2(x) \equiv 0$ satisfies condition (11), (12); hence the function w(x) is non-negative on [a,b]. We now show that w(x) is actually positive on [a,b].

If w is zero at an interior point, then w' is also zero there, otherwise w would have some negative values. But the identically zero function satisfies the same initial conditions as w(x) and by Theorem 5 for uniqueness of initial value problems, $w \equiv 0$ on [a,b] which contradicts (22). Hence w cannot vanish at an interior point.

If w vanishes at an endpoint, say at a, then from the first condition of (22) we have

$$w'(a) \cos\theta = -1 .$$

Now w non-negative implies that $w'(a) \geq 0$, a contradiction since $\cos\theta \geq 0$. Thus $w(a) > 0$. Similarly $w(b) > 0$, and we have $w > 0$ on [a,b].

As with Theorem 1 and Corollary 1, it is often possible to prove a more general theorem and by a judicious choice of parameters obtain a related, more specific result. For example, if $h \leq 0$, $0 \leq \theta \leq \pi/2$, $0 \leq \phi \leq \pi/2$; then the function $w \equiv 1$ satisfies (13) and (14) and Theorem 10 follows immediately from Theorem 11.

The fact that the function w does not appear in the inequality

$$z_2(x) \leq u(x) \leq z_1(x)$$

prompts us to try to obtain a necessary and sufficient condition that guarantees that the functions z_1 and z_2 form upper and lower bounds for u independent of a function w.

<u>Theorem 12</u>: Suppose that the functions $z_1(x)$ and $z_2(x)$ satisfy inequalities (9), (10) and (11), (12) respectively, in such a way that equality does not hold in all the conditions. Let g(x) be bounded below on each interval [a,c] and bounded above on each interval [c,b] with $a < c < b$. Let u(x) be a solution of (7), (8). Then the bounds

$$z_2(x) \leq u(x) \leq z_1(x) \tag{23}$$

hold if any only if $z_2(x) \leq z_1(x)$ for $a \leq x \leq b$.

Proof: Trivially, if (23) holds, then $z_2(x) \leq z_1(x)$. Hence assume $z_1 - z_2$ is non-negative and we must show that (23) holds. Put

$$q(x) \equiv z_1(x) - z_2(x)$$

If $q > 0$ on [a,b], then q satisfies the conditions for the positive function w in Theorem 11; i.e.,

$$(L+h)[q] = (L+h)[z_1] - (L+h)[z_2] \leq 0 \text{ on } (a,b) \tag{24}$$

since z_1 and z_2 satisfy (9) and (11) respectively.

$$\begin{aligned} -[z_1'(a) - z_2'(a)]\cos\theta + [z_1(a) - z_2(a)]\sin\theta &\geq 0 \\ [z_1'(b) - z_2'(b)]\cos\phi + [z_1(b) - z_2(b)]\sin\phi &\geq 0 \end{aligned} \tag{25}$$

since z_1 and z_2 satisfy (10) and (12) respectively. Hence we may conclude by Theorem 11 that

$$z_2(x) \leq u(x) \leq z_1(x).$$

Thus, we need only consider the case where q has a zero in [a,b]. We will show that q cannot vanish at an interior point and that if q vanishes at one or both endpoints then we may find suitable positive functions w(x) such that Theorem 11 will hold.

Suppose that $q(c) = 0$ for c in (a,b). Then $q'(c) = 0$ since $q \geq 0$ on [a,b], i.e., q has a local minimum at c. We may conclude that $q \equiv 0$ on [a,b] by the uniqueness of solutions to the initial value problem. But then equality holds in (24) and (25), a contradiction.

We are left with $q > 0$ interior to [a,b] and q vanishing at an end-point, say at $x = a$. Then $q'(a) > 0$ by above. Thus $\theta = \pi/2$, otherwise the first condition of (25) would be violated. Similarly if $q(b) = 0$, then $q'(b) < 0$ and $\phi = \pi/2$.

We now consider the case where $q(b) = 0$ and $q(a) > 0$, and proceed to find a positive function w such that Theorem 11 will hold. From the preceeding paragraph $\phi = \pi/2$. Consider the initial value problem

$$(26) \quad \begin{cases} (L+h)[r] = 0 \\ r(a) = \cos\theta \quad \text{and} \quad r'(a) = \sin\theta \end{cases}$$

which has a solution by classical existence Theorems for differential equations. Let r be the desired (unique) solution. We note that for $\theta < \pi/2$, r is (strictly) positive at a, and if $\theta = \pi/2$, $r(a) = 0$ and r is positive near a.

Now define the function $v = r/q$ for $a < x < b$. Also note that

$$v(a) = \frac{r(a)}{q(a)} \geq 0$$

and

$$v'(a) = \frac{q(a)\,r'(a) - q'(a)\,r(a)}{q^2(a)}$$

$$= \frac{q(a)\sin\theta - q'(a)\cos\theta}{q^2(a)}$$

by the first condition of (25). Using the facts that $(l+h)\,[r] = 0$ and $(L+h)\,[q] \quad 0$, we may conclude that v satisfies

$$(\bar{L}+H)[v] = v'' + Gv' + Hv = 0$$

with $G = (2q'/q) + g$ and $H = 1/q(L+h)[q] \leq 0$, as in the proof of Theorem 11. The coefficients G and H satisfy the restrictions in Theorems 2 and 3 on any subinterval of the form $[a,c]$. Hence either: (i) $v(c) > v(a)$ and $v'(c) > 0$, or (ii) $v(x) \equiv v(a)$ for x in $[a,c]$. In either case $r(x) > 0$ for x in $[a,b)$ since both v and q are positive.

If $v(x)$ is identically the positive constant $r(a)$ on $[a,b]$, then q is proportional to r and hence must satisfy the same equations. Thus

$$(L+h)[q] = 0$$

and

$$-r'(a)\cos\theta + r(a)\sin\theta = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$$

implies that

$$-q'(a)\cos\theta + q(a)\sin\theta = 0.$$

By hypothesis on z_1 and z_2 (hence q) equality does not hold in all conditions (24) and (25). We must then have

$$q'(b)\cancelto{0}{\cos\phi} + q(b)\cancelto{1}{\sin\phi} > 0$$

from the second condition of (25) and that $\phi = \pi/2$. Hence $q(b) > 0$, a contradiction. Thus we are left with case (i), i.e., for some number c in (a,b),

$$v(c) > v(a) \geq 0 \quad \text{and} \quad v'(x) > 0 \quad \text{for } x \geq c.$$

Now

$$r'(x) = \frac{r'q - rq'}{q^2}$$

Put $\psi(x) \equiv r'q - rq'$. Then from (24) and (26)

$$(27) \quad -r(L+h)[q] = -rq'' - rgq' - rhq \geq 0$$

and

$$(28) \quad q(L+h)[r] = qr'' + qgr + qhr = 0.$$

Solving for -qhr in (28) and substituting into (27) we see that satisfies the differential inequality $\psi' + g\psi = -r(L_1+h)[q] \geq 0.$

On the interval [c,b] $g \leq M$ by hypothesis, thus

$$\frac{d}{dx} \ln \psi = \frac{\psi'}{\psi} \geq -g \geq -M .$$

Solving for ψ we may write

$$\psi(x) = Ae^{-M(x-c)}$$

and for x = c

$$\psi(c) \geq A ,$$

thus

$$\psi(x) \geq \psi(c)e^{-M(x-c)}$$

In particular

$$\psi(b) \geq \psi(c)e^{-M(x-c)} > 0$$

since v(c) > 0. i.e.,

$$r'(b)\cancel{q(b)}^{\,0} - q'(b)r(b) \geq 0.$$

But q(b) = 0 and q'(b) $\leq$ 0, therefore r(b) > 0. Hence we have r positive on (a,b] and satisfying

$$(L_1+h)[r] = 0$$

with

$$-r'(a)\cos\theta + r(a)\sin\theta = 0$$

and since $\phi = \pi/2$

$$r'(b)\cos\phi + r(b)\sin\phi = r(b) > 0.$$

Hence the function

$$w(x) \equiv q(x) + r(x)$$

satisfies the requirements of Theorem 11, and we have

$$z_2(x) \leq u(x) \leq z_1(x).$$

On the other hand, if q(a) = 0 and q(b) $>$ 0, analogously we may show that the solution of

$$\begin{cases} (L+h)[s] = 0 \\ s(b) = \cos\phi \quad \text{and} \quad s'(b) = -\sin\phi \end{cases}$$

is positive on [a,b), so that the function

$$w(x) \equiv q(x) + s(x)$$

satisfies Theorem 11.

Finally if q(a) = q(b) = 0, we have r $>$ 0 on (a,b] and s $>$ 0 on [a,b) with r(a) = s(b) = 0; hence

$$w(x) \equiv r(x) + s(x)$$

satisfies Theorem 11. This concludes the proof of Theorem 12.

VI. APPLICATIONS OF APPROXIMATION FUNCTIONS

Through the preceeding sections we have seen that the results have been essentially theoretical and that examples have not always been fruitful or plentiful. Sections 1 - 4 present one-dimensional maximum principles and their use in some differential equations theory while Section 5 actually gives some practical techniques to apply in boundary value problems. The problems we are considering are second-order differential equations with variable coefficients which in general must be solved by series or approximation methods.

There does not seem to be a "best" form for an approximation function since, in general, each boundary value problem needs individual attention. However, there is a general procedure to follow to determine bounds and behavior patterns of the solution (under certain conditions) and then the approximate value of the solution at any point in the interval under consideration:

1. Find the straight line $z(x)$ which satisfies the boundary conditions as equalities and apply the operator $(L+h)$ to it. If $(L+h)[z]$ is of one sign on the interval, then z will serve as either an upper or lower bound as determined by the sign of $(L+h)[z]$. (i.e., $(L+h)[z] \leq 0$ implies that $u(x) \leq z(x)$ and $(L+h)[z] \geq 0$ implies that $z(x) \leq u(x)$).

2. If the constants γ_1, γ_2 of the boundary conditions are of the same sign, then the approximation function $z(x) \equiv 0$ will show that the solution $u(x)$ is either non-negative or non-positive. (i.e., $\gamma_1 \geq 0$, $\gamma_2 \geq 0$ implies $u \geq 0$ and $\gamma_1 \leq 0$, $\gamma_2 \leq 0$ implies $u \leq 0$).

3. If the general behavior of u(x) is desired, it is only necessary to bound u between the straight line from (1.) (assuming this is possible) and some constant function which satisfies the approximation boundary conditions (9) and (10), or (11) and (12) in Section V.

4. If an approximate value of the solution u(x) is desired at some point in the interval then find a polynomial of exponential function that satisfies the approximation boundary conditions mentioned in (3.) above. If a more exact approximation is desired, then the form of the approximation function is more complicated and the difficulty of evaluating and checking the restrictions increases. In many instances it would be as easy to find an approximate or series solution (if initial conditions are known) and evaluate it numerically on a computer. The value of the above technique is that the behavior of the solution of a complicated second order differential equation can be determined easily and in less time than other methods.

The following examples will help point out the applications of Section 5 and Theorem 12 specifically. The choice of the approximation functions may seem ambiguous, but they readily become more understandable in the context of each problem.

Example 1: Consider the boundary value problem

$$(1) \qquad u'' + 3u' - xu = 0 \quad \text{on } (0,1)$$

with

$$(2) \qquad -u'(0) + u(0) = 0 \quad \text{and} \quad u(1) = 1.$$

As noted in (2.) of the preceeding discussion the solution of (1) satisfying (2) is non-negative. Consider the straight line $z(x) = \frac{1}{2}(x+1)$

which satisfies (2). Then

$$(L+h)[z] \equiv z'' + 3z' - xz$$
$$= \frac{1}{2}\left[3 - x(1+x)\right] \geq 0$$

on (0,1). Hence let $z(x) = z_2(x)$ and we have

$$\frac{1}{2}(x+1) \leq u(x)$$

We note that $y \equiv 1$ satisfies the requirements for an upper bound, i.e., $(L+h)[y] \leq 0$ and $y(1) = 1$, $-y'(0) + y(0) = 1 > 0$. We have quite simply determined the general behavior of u in (0,1), i.e.,

$$(1/2)(x+1) \leq u(x) \leq 1.$$

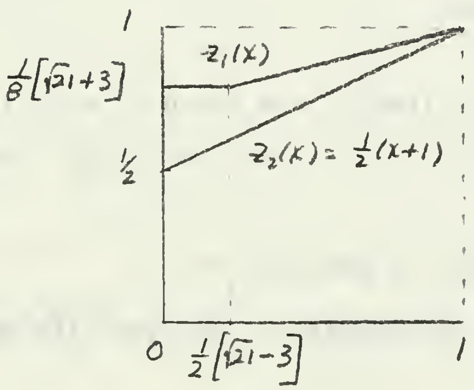


Fig. 2

To determine more accurately the bounds for u at any point x, we choose a more complicated function for z_1, the upper bound. A natural choice to consider would be

$$z_1(x) = (1/4)(x+3)$$

which satisfies

$$z(1) = 1 \text{ and } -z'(0) + z(0) = 3/4 > 0.$$

But z(x) does not satisfy

$$(3) \quad (L+h)[z_1] = \frac{1}{4}\left[3 - x(x+3)\right] \leq 0$$

on the entire interval (0,1). But we may define

$$z_1(x) = \begin{cases} \frac{1}{8}\left[\sqrt{21}+3\right] ; & 0 < x \le \frac{1}{2}\left[\sqrt{21}-3\right] \\ \frac{1}{4}(x+3) ; & \frac{1}{2}\left[\sqrt{21}-3\right] < x \le 1 \end{cases}$$

where $(1/2)(\sqrt{21}-3)$ is a zero of $3 - x(x+3) = 0$. Then z_1 is continuous and satisfies the conditions for an upper bound and we have

$$\frac{1}{2}(x+1) \le u(x) \le \frac{1}{8}\left[\sqrt{21}+3\right]$$

on $(0,(1/2)(\sqrt{21} - 3))$ and

$$\frac{1}{2}(x+1) \le u(x) \le \frac{1}{4}(x+3)$$

on $((1/2)(\sqrt{21} - 3),1)$.

<u>Example 2</u>: Consider

(4) $$u'' - (1+x^2)u = 0 \text{ for } 0 < x < 1$$

with

(5) $$u(0) = 1 \text{ and } u(1) = 0.$$

The solution u must be non-negative. Consider the straight line

$$z_1(x) = 1-x$$

which satisfies (5). Now

$$(L+h)[z_1] = z_1'' - (1+x^2)z_1$$
$$= 0 - (1+x^2)(1-x) \le 0$$

since $0 < x < 1$. We now have an approximate idea of the shape of u. We pick for z_2 (the lower bound) a parabolic function. We choose z_2 such that $z_2(0) = 1$ and $z_2(1) = 0$, and so that z_2 is concave upward. In Example 1 we showed steps 1-3 of the general method and here we show how a better bound can be found by choosing a more complicated approximation function. Let

$$z_2(x) = (x-1)^2$$

which satisfies (5). Then

$$(L+h)[z_2] = z_2'' - (1+x^2) z_2$$

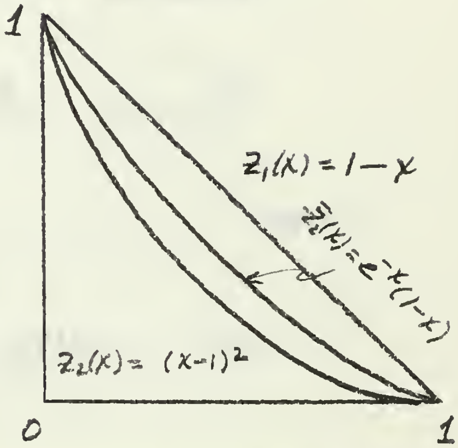
$$= 2 - (1+x^2)(x-1)^2 \geq 0 \quad \text{on } (0,1).$$

Hence

$$(x-1)^2 \leq u(x) \leq 1-x$$

In particular x = 1/2

$$\frac{1}{4} \leq u(\tfrac{1}{2}) \leq \tfrac{1}{2}.$$



Choosing for $\bar{z}_2$ the expotential function

$$\bar{z}_2(x) = e^{-x}(1-x)$$

which satisfies

$$\bar{z}_2(0) = 1 \quad \text{and} \quad \bar{z}_2(1) = 0.$$

Now

$$(L+h)[\bar{z}_2] \equiv \bar{z}_2'' - (1+x^2)\bar{z}_2$$

$$= (3-x)e^{-x} - (1+x^2)(1-x)e^{-x}$$

$$= e^{-x}[x^3 - x^2 + 2] \geq 0 \quad \text{on } (0,1)$$

since $e^{-x} > 0$ and $2 > x^3 - x^2$. Hence

$$e^{-x}(1-x) \leq u(x) \leq 1-x,$$

and in particular at x = 1/2

$$.303 \leq u(x) \leq .500,$$

a considerable improvement.

Since $z_1(0) = \bar{z}_2(0) = u(0)$, we may bound $u'(0)$. Now

$$u'(0) = \lim_{x \to 0} \left[\frac{u(x) - u(0)}{x} \right],$$

and from our previous results

$$-2 = \bar{z}_2'(0) \le u'(0) \le z_1'(0) = -1.$$

Example 3: Consider the problem

(6) $$u'' - xu = 0 \quad for \ 0 < x < 1$$

with

(7) $$-u'(0) + u(0) = 0 \quad and \quad u(1) = 1.$$

Noting the similarity of boundary conditions (7) with boundary conditions (2) in Example 1, we choose

$$z(x) = \frac{1}{2}(x+1)$$

and note that

$$(L+h)[z] = z'' - xz$$
$$= -\frac{1}{2}x(x+1) \le 0 \quad on \ (0,1).$$

Hence $z_1(x) = (1/2)(x+1)$ is an upper bound for the solution $u(x)$. We choose for z_2 the exponential function

$$z_2(x) = e^{x-1}$$

which satisfies (7). Then

$$(L+h)[z_2] = (1-x)e^{x-1} \ge 0 \quad on \ (0,1).$$

Thus

$$e^{x-1} \le u(x) \le \frac{1}{2}(x+1),$$

and in particular at $x = 1/2$,

$$0.6065 \leq u(1/2) \leq 0.7500.$$

An interesting application of the theory developed allows us to bound solutions to the differential equation

$$(L+h)[u] \equiv u'' + g(x)u' + h(x)u = f(x)$$

on (a,b) even when the coefficients $g(x)$ and $h(x)$ are piece-wise continuous.

<u>Example 4</u>: Consider

$$u'' + xu' - h(x)u = 0 \quad \text{for } 0 < x < 1 \tag{8}$$

with

$$u(0) = 0 \quad \text{and} \quad u(1) = 1 \tag{9}$$

where

$$h(x) = \begin{cases} 1 \; ; \; 0 < x \leq 1/2 \\ 2 \; ; \; 1/2 < x \leq 1 \end{cases}$$

From the boundary conditions consider the approximation function

$$z(x) = x$$

which satisfies (9). Then

$$(L+h)[z] = z'' + xz' - h(x)z$$

$$= \begin{cases} x - x = 0 \; ; \; 0 < x \leq 1/2 \\ x - 2x = -x < 0 \; ; \; 1/2 < x \leq 1. \end{cases}$$

Hence $0 \leq u(x) \leq x$, since u must be non-negative by (9). A better lower bound may be determined by considering the parabola

$$z_2(x) = x^2$$

which satisfies (9) and

$$(L+h)[z_2] = z_2'' + x z_2' - h(x) z_2$$

$$= \begin{cases} x^2+2 \geq 0 \; ; \; 0 < x \leq \frac{1}{2} \\ 2 \; ; \; \frac{1}{2} < x \leq 1 \; . \end{cases}$$

Thus

$$x^2 \leq u(x) \leq x.$$

When working through the theoretical background for this technique, there is a question of generalizing the approximation functions so that the "best" bound for the solution u can be obtained. In considering this, the following discussion will point out some problems which arise.

We will choose general expressions containing parameters for the approximating functions z_1 and z_2 and attempt to pick values of the parameters to make z_1 and z_2 satisfy the appropriate conditions. For example choose

$$z_1(x) = A\left[2 - e^{-\alpha(x-a)}\right]$$

and

$$z_2(x) = B\left[2 - e^{-\alpha(x-a)}\right]$$

where A,B and α are constants to be determined. From Section V we know that $z_1(x)$ must satisfy

$$(L+h)[z_1] \leq f(x) \quad \text{on } (a,b)$$

and

$$z_1(a) \geq \gamma_1 \quad ; \quad z_1(b) \geq \gamma_2 \; .$$

Then

$$(L+h)[z_1] = z_1'' + g(x) z_1' + h(x) z_1$$
$$= -Ae^{-\alpha(x-a)}\left[\alpha^2 - \alpha g(x) + h(x)\right] + 2Ah(x).$$

Now choose α so large that

$$\alpha^2 - \alpha g(x) + h(x) > 0,$$

where f,g and h are assumed to be bounded on [a,b]. To make $z_1(x)$ an upper bound, we must have

$$(L+h)[z_1] = 2A\,h(x) - A\left(\alpha^2 - \alpha g(x) + h(x)\right)e^{-\alpha(x-a)} \leq f(x),$$

or

$$A\left[2h(x) - (\alpha^2 - \alpha g(x) + h(x))e^{-\alpha(x-a)}\right] \leq f(x).$$

This is satisfied if we let

$$k = \min_{a \leq x \leq b}\left[-2h(x) + (\alpha^2 - \alpha g(x) + h(x))e^{-\alpha(x-a)}\right],$$

and then if we choose A such that

$$A \geq \frac{1}{k}\max_{0 \leq x \leq 1}\{-f(x)\}.$$

By its definition $k > 0$. We have now satisfied the condition

$$(L+h)[z_1] \leq f(x) \quad \text{for} \quad a < x < b.$$

We must also satisfy

$$z_1(a) = A\left[2 - e^{-\alpha(a-a)}\right] = A \geq \gamma_1,$$

$$z_2(b) = A\left[2 - e^{-\alpha(b-a)}\right] \geq \gamma_2.$$

Let $c = 2 - e^{-\alpha(b-a)}$, and choose A such that

$$A = \max\left[\gamma_1,\ \frac{\gamma_2}{c},\ \frac{1}{k}\max_{0 \leq x \leq 1}\{-f(x)\},\ 0\right]$$

i.e., A is the largest of the four numbers in the brackets.

Then

$$u(x) \leq z_1(x).$$

Using the same α in $z_2(x)$, we choose B as the smallest of the four numbers

$$B = \min\left[\gamma_1, \frac{\gamma_2}{c}, \min_{0 \le x \le 1}\{-f(x)\}, 0\right],$$

then

$$B\left[2-e^{-\alpha(x-a)}\right] \le u(x) \le A\left[2-e^{-\alpha(x-a)}\right]$$

on $[a,b]$.

Example 5: We solve the following problem in two ways. Consider

$$u'' - xu = 0 \text{ for } 0 < x < 1$$

with

$$u(0) = 0 \text{ and } u(1) = 1.$$

Let $z_1(x) = x$ which satisfies the boundary conditions, and

$$(L+h)\quad [z_1] = -x^2 \le 0 \text{ on } (0,1).$$

Hence

$$0 \le u(x) \le x.$$

For z_2 we choose

$$z_2(x) = x - Bx(1-x)$$

which satisfies the boundary conditions, and $B > 0$. Then

$$(L+h)[z_2] = -x^2 + B\left[2 + x^2(1-x)\right] \ge 0 \quad \text{on } (0,1)$$

for $B \ge 1/2$, which can be seen by considering the values of B which make

$$(L+h)[z_2]\Big|_{x=1} = 0$$

since

$$(L+h)[z_2]\Big|_{x=0} > 0,$$

and we are dealing with a cubic equation. Hence we choose B = 1/2, and we have

$$(1/2)(x+x^2) \leq u(x) \leq x,$$

since $z_2 \leq z_1$ on (0,1). From the theory developed before this problem, choose the approximation functions as follows:

$$z_1(x) = A\left[2 - e^{-\alpha x}\right]$$

and

$$z_2(x) = B\left[2 - e^{-\alpha x}\right]$$

Then we must choose α so large that

$$\alpha^2 - x > 0 \qquad \text{on } (0,1),$$

hence choose $\alpha = 1$. Then

$$k = \min_{0 \leq x \leq 1}\left[2x + (1-x)e^{-x}\right] = 1 \quad \text{(at } x = 0\text{)},$$

and

$$c = 2 - e^{-1}.$$

Hence

$$A = \mathrm{Max}\left[0, \frac{1}{2-e^{-1}}, 0, 0\right] = \frac{e}{2e-1}$$

and

$$B = \min\left[0, \frac{1}{2-e^{-1}}, 0, 0\right] = 0.$$

Hence

$$0 \leq u(x) \leq \frac{e}{2e-1}\left[2 - e^{-x}\right]$$

which does not improve on our previous results.

The method outlined previously and illustrated in the example also leads to very interesting aspects of bounding solutions of second-order boundary value problems with general boundary conditions.

We may find the straight line

$$z(x) = c_1 x + c_2 \tag{10}$$

which satisfies the general boundary conditions:

$$\left\{\begin{aligned} -z'(a)\cos\theta + z(a)\sin\theta &= \gamma_1 \\ z'(b)\cos\phi + z(b)\sin\phi &= \gamma_2 \end{aligned}\right. \tag{11}$$

if the parameters θ and ϕ satisfy certain restrictions.

We may rewrite (11) as

$$c_1(-\cos\theta) + (c_1 a + c_2)\sin\theta = \gamma_1$$

$$c_2\cos\phi + (c_1 b + c_2)\sin\phi = \gamma_2$$

or equivalently

$$c_1\left[a\sin\theta - \cos\theta\right] + c_2\sin\theta = \gamma_1$$

$$c_1\left[b\sin\phi + \cos\phi\right] + c_2\sin\phi = \gamma_2\,,$$

which has a non-trival solution for c_1 and c_2 if the determinant of co-efficients

$$D = \begin{bmatrix} (a\sin\theta - \cos\theta) & \sin\theta \\ (b\sin\phi + \cos\phi) & \sin\phi \end{bmatrix}$$

is non-zero (assuming $\gamma_1^2 + \gamma_2^2 \neq 0$).

$$D = (a-b)\sin\theta\sin\phi - \left[\sin\theta\cos\phi + \cos\theta\sin\phi\right]$$

$$= (a-b)\sin\theta\sin\phi - \sin(\theta+\phi).$$

Hence if θ and ϕ are not solutions of

$$\sin(\theta+\phi) = (a-b)\sin\theta\sin\phi \tag{12}$$

then we may always find the line (10). We note that if $\theta + \phi = n\pi$ and $\theta = n\pi$, $\phi = n\pi$ for (n = 0, 1,...), then (12) is satisfied.

In Example 1 we used a polygonal line as an upper bound. This procedure may not be fruitful in the non-homogeneous case. The reason for this will be made clear later.

Example 6: Consider the non-homogeneous boundary value problem

$$u'' - u = 2 + x - x^2$$

with

$$u(0) = u(1) = 0 .$$

To determine possible concavity of the solution, consider

$$z(x) \equiv 1$$

then

$$(L+h)[z] \equiv z'' - z = -1 < 2 + x - x^2 \quad \text{on } (0,1);$$

hence z $\equiv$ 1 is an upper bound and the solution u(x) is concave upward since

$$z'' = -1 + z < 0 \quad \text{on } (0,1).$$

To determine a lower bound, consider

$$z(x) \equiv 3$$

then

$$(L+h)[z] \equiv z'' - z = 3 > 2 + x - x^2 \quad \text{on } (0,1);$$

hence z $\equiv$ -3 is a lower bound since the appropriate endpoint inequalities hold.

By substitution we may verify that

$$u(x) = x^2 - x$$

is the solution. We might expect then that the polygonal line

$$z_2(x) = \begin{cases} -x\ ; & 0 \le x \le 1/4 \\ -1/4\ ; & 1/4 < x \le 3/4 \\ x-1\ ; & 3/4 < x \le 1 \end{cases}$$

would satisfy the theoretical conditions for a lower bound since

$z_2(x) \leq u(x)$ on $(0,1)$.

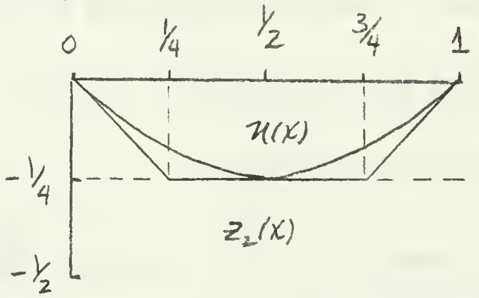


Fig. 4

Note that this is the opposite of the procedure in our theory where we determine bounding functions by applying the restrictions to them rather than determining whether actual bounds of the solution satisfy the restrictions. Recall that for z_2 to be a lower bound, then

(i) $(L+h)\ \ [z_2] \geq f(x) = 2 + x - x^2$ on $(0,1)$

(ii) $z_2(0) \leq 0$ and $z_2(1) \leq 0$.

Applying the operator $(L+h)$ to z_2,

$$(L+h)[z_2] \equiv z_2'' - z_2 = \begin{cases} x\ ; & 0 \le x \le 1/4 \\ 1/4\ ; & 1/4 < x \le 3/4 \\ 1-x\ ; & 3/4 < x \le 1 \end{cases}$$

$$\leq 2 + x - x^2 \quad \text{on} \quad (0,1),$$

but

$$z_2(0) = 0 \text{ and } z_2(1) = 0.$$

From our theory this would seem to imply that z_2 is an upper bound for u; but by an oversight we have not considered the fact that each segment of the polygonal line must satisfy the restrictions individually.

Hence for $z_2(x) = -x$ $[0,1/4]$ we see that

$$(L+h)[z_2] = x < f(x)$$

with

$$z_2(0) = 0 \quad \text{and} \quad z_2(\tfrac{1}{4}) = -\tfrac{1}{4} \leq 0.$$

Thus our theory does not apply. We find similar results for the remaining segments of z_2.

This points out an important result that for the non-homogeneous case the theory developed may not apply to the polygonal line approximation functions.

The reason for this is clear upon consideration of Example 1, where we choose the straight line

$$z(x) = \frac{1}{4}(x+3)$$

as an upper bound, and find that

$$(L+h)[z] \equiv z'' + 3z' - xz = \frac{1}{4}[3 - x(x+3)]$$

changes sign on $(0,1)$, i.e., $(L+h)[z]$ has a zero in $(0,1)$ and by suitably changing $z(x)$ we can satisfy the restrictions

$$(L+h)[z] \leq 0 \text{ on } (0,1).$$

But for the non-homogeneous case, we must satisfy

$$(L+h)[z] \leq f(x)$$

and we may not be able to suitably change our approximation function to satisfy the conditions.

This points out the importance of parametric approximation functions and being able to choose the parameters in such a manner as to satisfy the restrictions.

Consider then

$$z_1(x) = A\left[2 - e^{-\alpha x}\right]$$

$$z_2(x) = B\left[2 - e^{-\alpha x}\right]$$

and choose A, B and α according to the theory developed preceeding Example 5 so that z_1 and z_2 are upper and lower bounds, respectively.

Then α must satisfy $\alpha^2 - 1 > 0$; hence, choose $\alpha = 2$. Then choose

$$k = \min_{0 \le x \le 1}\left[2 + 3e^{-2x}\right] = 2 + 3e^{-2} \quad at \quad x = 1.$$

We must now determine

$$\max_{0 \le x \le 1}\{-f(x)\} = \max_{0 \le x \le 1}\{x^2 - x - 2\} = -2 \ at\ x = 0 \ \text{or}\ x = 1;$$

and

$$\min_{0 \le x \le 1}\{-f(x)\} = \min_{0 \le x \le 1}\{x^2 - x - 2\} = -\frac{9}{4} \quad at \quad x = \tfrac{1}{2}.$$

Since $\gamma_2 = 0$, we do not need $c = 2 - e^{-2}$ since $\gamma_2/c = 0$. Thus choose A and B so that

$$A = \max\left[0, 0, \frac{-2}{2 + 3e^{-2}}, 0\right] = 0$$

and

$$B = \min\left[0, 0, \frac{-9}{8 + 12e^{-2}}, 0\right] = -\frac{9}{8 + 12e^{-2}}$$

Thus

$$\frac{-9}{8 + 12e^{-2}}\left[2 - e^{-2x}\right] \le u(x) \le 0 .$$

We may graph the above by approximating the function $z_2(x)$, and we have

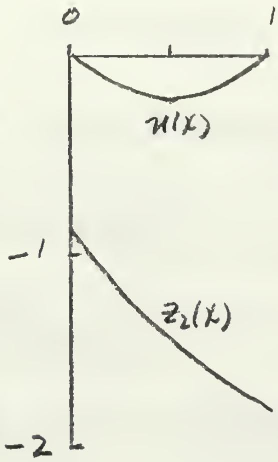


Fig. 5

We now have the option of attempting to change the parameters A, B and α to obtain better bounding functions $\bar{z}_1$ and $\bar{z}_2$ or keeping the existing functions and tolerating the possible large error. In either case we are sacrificing time or exactness and are defeating our purpose.

The technique used in the preceeding examples has indicated the possibility of approximating solutions u(x) by multiplying the straight line s(x) found above by "warping" functions ξ (x) and ζ (x) so that

$$\xi(x)s(x) \leq u(x) \leq \zeta(x)s(x)$$

This technique should not only help determine the general behavior of the solution but yield usable numerical approximations.

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13. ABSTRACT

The problem considered is the application of a one-dimensional maximum principle to second order, linear differential equations of the form

$$u'' + g(x)u' + h(x)u = f(x) \text{ for } a < x < b$$

with associated general boundary conditions to obtain functions $z_1(x)$ and $z_2(x)$ such that

$$z_2(x) \leq u(x) \leq z_1(x)$$

on [a,b]. The functions f, g and h are assumed to be bounded. We wish to determine the behavior of the solution u(x) on [a,b] and also to obtain reliable numerical estimates of u.

The basic concepts in the theoretical background are expanded versions of a presentation in Protter and Weinberger [Ref. 4].

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